XXII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The correspondence round. Solutions

1. (8) (S.Kuznetsov) Let triangles T and ABC be symmetric about a point P. Points A', B', C' are the reflections of A, B, C about the corresponding sidelines of T. It is known that two of these points coincide. Prove that all three points coincide.

   Solution. Since the corresponding sidelines of T and ABC are parallel, the points A', B', C' lie on the altitudes of triangle ABC. If, for example, A' and B' coincide, then the corresponding point is the orthocenter H of ABC, thus two sidelines of T are the perpendicular bisectors to AH and BH. Let A_0, B_0, A_1, B_1 be the midpoints of segments BC, AC, AH, BH respectively. Then $A_0B_0A_1B_1$ is a rectangle, because its sides are the medial lines of triangles ABC, ABH, ACH, BCH (fig. 1). The reflection about the center of this rectangle maps AC and BC to the perpendicular bisectors to BH and AH, hence it maps AB to the third side of T. But this reflection maps AB to the perpendicular bisector to CH, therefore C' also coincides with H.

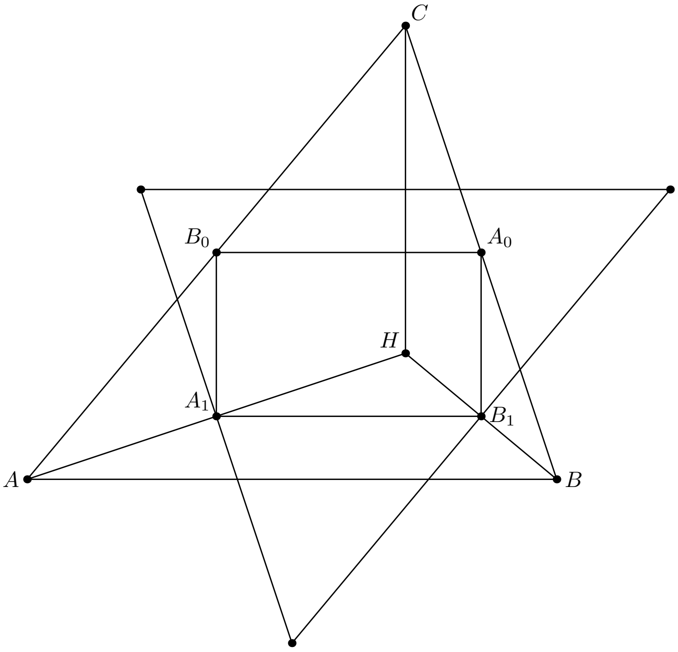

Fig. 1.

   Remark. The triangles ABC and T have a common nine-point-circle and are symmetric about its center.

2. (8) (S.Khalifah) An isosceles trapezoid $ABCD$ is circumscribed around a circle touching the lateral side AB at point T. The segments TC and TD meet this circle for the second time at points P, Q respectively. Find all possible values of $TP/PC + TQ/QD$.

   Answer. 8.

   Solution. Let M, N be the midpoints of bases BC, AD respectively (fig. 2). Then $BT^2 = CM^2 = CP \cdot CT = CP(CP + TP)$, therefore,

$$\frac{TP}{CP} = \frac{CM^2}{CP^2} + 1 = \frac{CT^2}{BT^2} + 1.$$

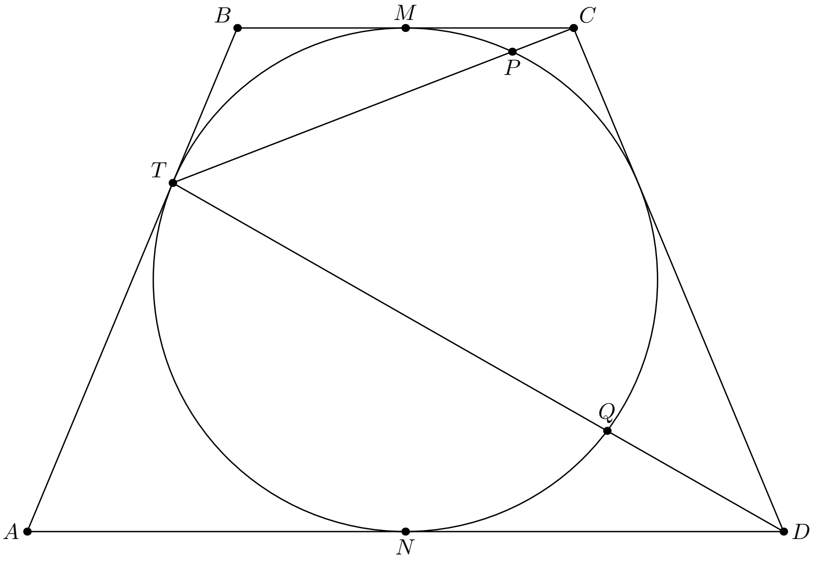

Fig. 2.

Applying the cosines law to the triangle CBT with $BC = 2BT$, we obtain $CT^2 = BT^2(5 - 4\cos\angle B)$. Similarly $DT^2 = AT^2(5 - 4\cos\angle A)$. Since $\cos\angle A + \cos\angle B = 0$, we obtain that $CT^2/BT^2 + DT^2/AT^2 = 10$ and $TP/PC + TQ/QD = 8$.

3. (8) (I.Kukharchuk) Let AK be the bisector of a triangle ABC, N be a point on AC such that $\angle NKC = \angle CAB/2$, and L be the midpoint of KN. Prove that $\angle KBN = \angle LAK$.

   Solution. Since $\angle NKC = \angle CAK = \angle KAB$, we have $\angle AKB = \angle ANK$ (fig. 3). Therefore the triangles ANK and AKB are similar and $AK : AN = BK : KN$. Applying the sines law to the triangles ANL, ALK, and NKB, we obtain that $\sin\angle NAL : \sin\angle LAK = \sin\angle BNK : \sin\angle KBN$, and since $\angle BNK + \angle KBN = \angle NAK$, this yields the required equality.

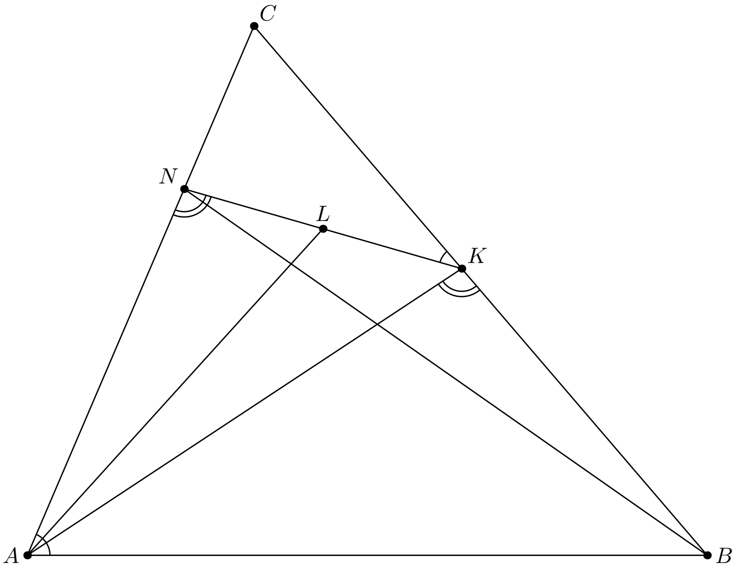

Fig. 3.

4. (8) (G.Kuznetsov) The diagonals of a circumscribed quadrilateral $ABCD$ meet at point X. Prove that there exists a common tangent to the incircles of triangles ABC, BCD, and AXD.

   Solution. Let l_1 be the length of the common external tangent to the incircles of triangles ABC, BCD; l_2 be the length of the common internal tangent to the incircles of triangles ABC, AXD; l_3 be the length of the common internal tangent to the incircles of triangles BCD, AXD. We have

$$l_1 = \frac{BC + AC - AB}{2} - \frac{BC + CD - BD}{2} = \frac{(AC + BD) - (AB + CD)}{2},$$

$$l_2 = \frac{AC + AB - BC}{2} - \frac{AX + AD - DX}{2}, \quad l_3 = \frac{BD + CD - BC}{2} - \frac{AD + DX - AX}{2},$$

$$l_2 + l_3 = \frac{(AC + BD) + (AB + CD) - 2(AD + BC)}{2}.$$

   Since $AB + CD = AD + BC$, we obtain that $l_1 = l_2 + l_3$, therefore the common external tangent to the incircles of triangles ABC and BCD, distinct from BC, touches also the third incircle (fig. 4).

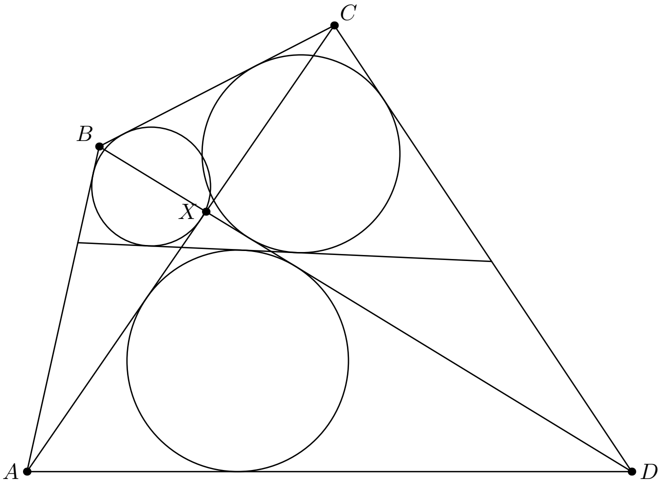

Fig. 4.

5. (8) (Ya.Scherbatov) Let I be the incenter of a triangle ABC. The perpendicular bisector to AI meets BC at point D; the line AD meets for the second time the circumcircle of ABC at point X. Prove that $|BX - CX| = AX$.

   Solution. Let B_0, C_0 be the midpoints of arcs AC, AB respectively; and P be the second common point of the circle DBB_0 and the line AD (fig. 5). Then $\angle BPX = \angle BB_0C_0 = (180^\circ - \angle BXP)/2$, $\angle B_0PX = \angle B_0BC = \angle B_0XP$, therefore $XP = XB$, $B_0X = B_0P$, and the triangles AB_0P, CB_0X are congruent. Hence $XC = AP$, $AX = XP - AP = XB - XC$.

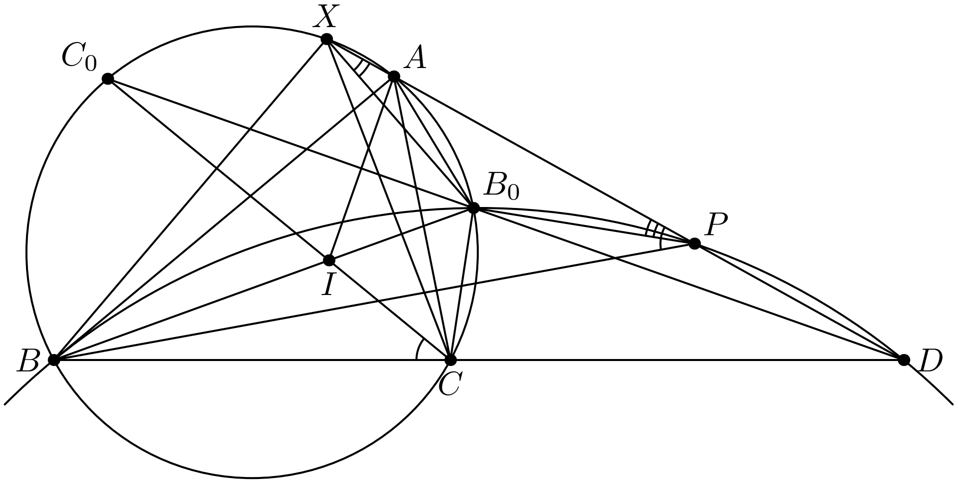

Fig. 5.

6. (8–9) (L.Emelyanov) Let O be the circumcenter of a triangle ABC, I be its incenter, H be the orthocenter, and N be the Nagel point. Prove that $IN = IH$ if and only if ONH is a right angle.

   Solution. Let M be the centroid of the triangle and E be the midpoint of OH (the center of the nine-points-circle). Then the segments IN and OH meet at M, and $OM : MH = IM : MN = EM : MO = 1 : 2$ (fig. 6). Two last equalities yield that $IE \parallel ON$, thus IE bisects the segment NH. Hence $IN = IH$ iff $IE \perp NH$, i.e. the angle ONH is right.

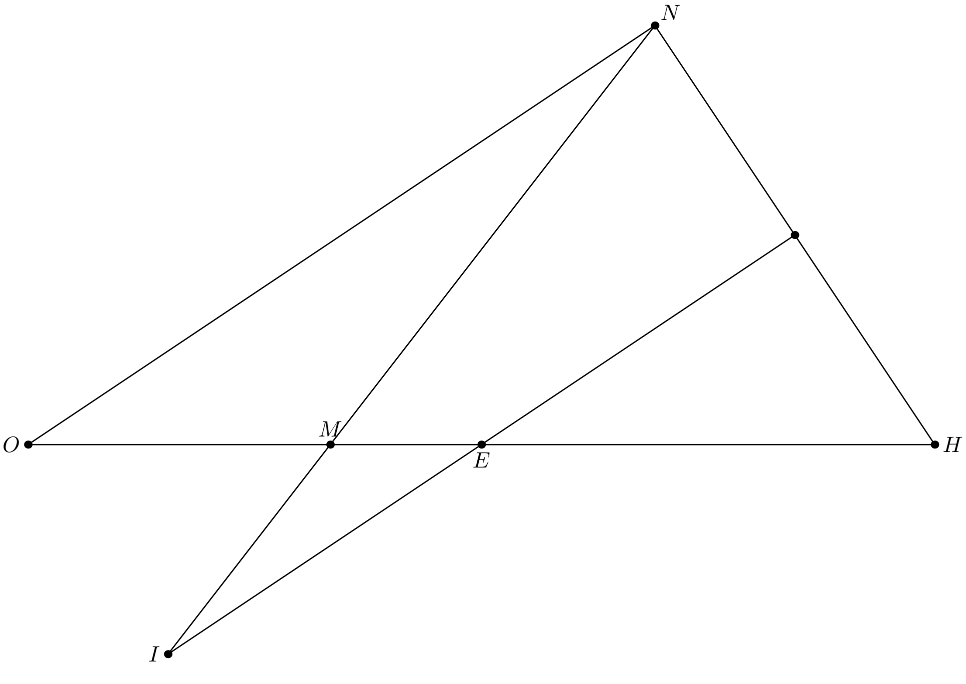

Fig. 6.

7. (8–9) (A.Zaslavsky) The side AB of a triangle ABC touches the incircle and the excircle at points P and Q respectively. Let T be the projection of the midpoint of AB to the bisector of angle C. Prove that C, P, Q, T are concyclic.

First solution. Let M be the midpoint of AB, I be the incenter of the triangle, K be the point of the incircle, opposite to P, and L be the foot of the bisector from C (fig. 7). Then K lies on CQ, and $\angle KIC = \angle PMT = |\angle A - \angle B|/2$. Also $MP = |a-b|/2$, $ML = |a-b|c/(2(a+b))$, where a, b, c are the sidelengths of the triangle. Hence

$$MT : MP = c\cos\frac{A-B}{2} : (a+b) = \sin C\cos\frac{A-B}{2} : (\sin A+\sin B) = \sin C : 2\sin\frac{A+B}{2} = \sin\frac{C}{2}.$$

Since $\sin\frac{C}{2} = IK : IC$ the triangles MTP and IKC are similar. Therefore $\angle QPT = \angle QCT$, and $CPTQ$ is cyclic.

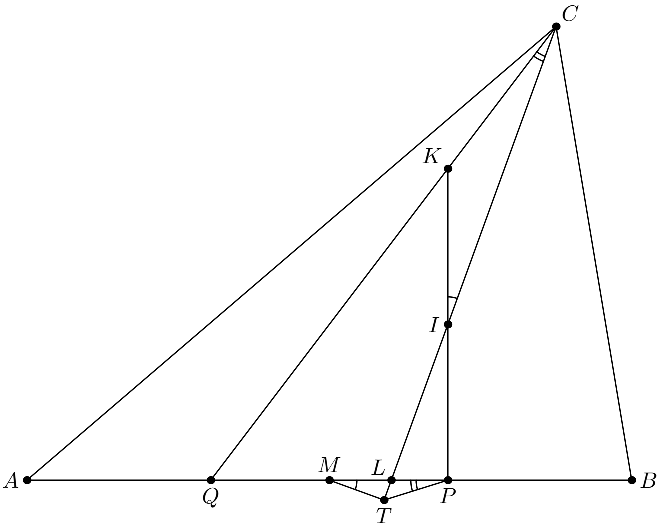

Fig. 7.

Second solution. Since $\angle IPM = \angle ITM = 90^\circ$, the points I, P, T, M are concyclic, i.e. $\angle PTC = \angle LMI$. Also $MP = MQ = |a-b|/2$, $ML = c|a-b|/2(a+b)$, therefore $MQ : LM = (a+b) : c = CI : IL$. Hence $MI \parallel CQ$, $\angle CQP = \angle CTP$, and C, P, Q, T are concyclic.

8. (8–9) (D.Krokhalev, G.Kuznetsov, A.Kovalenko) Let ABC be a triangle with $\angle B = 30^\circ$, O be the circumcenter of ABC, I be its incenter. The circles AIB, CIB meet BC, AB respectively at points D, E. Prove that D is the orthocenter of triangle OEI.

Solution. Since $\angle B = 30^\circ$, we obtain that $\angle AIC = 105^\circ$, $\angle AID = \angle CIE = 150^\circ$. Then $\angle AIE = \angle CID = 105^\circ$, i.e. D and E are the reflections of A, C about CI, AI respectively. Hence $AE = CD = AC = AO = CO$ and $\angle ODI = \angle ODC - \angle IDC = 90^\circ - \angle DOC/2 - \angle A/2 = 120^\circ - \angle C/2 - \angle A/2 = 45^\circ$ (fig. 8). But $\angle EID = 45^\circ$, therefore $OD \perp IE$. Similarly $OE \perp ID$.

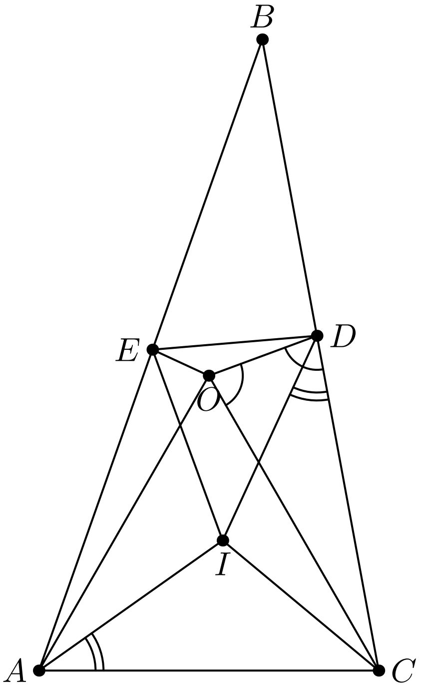

Fig. 8.

9. (8–9) (I.Kukharchuk) Let $ABCD$ be a circumscribed quadrilateral with incenter I. The circles BID and AIC meet at point P, and the rays AB and DC meet at point Q. Let R be the midpoint of PI. Prove that the quadrilateral $ARQD$ is cyclic.

   Solution. Let K, L, M, N be the touching points of the incircle with AB, BC, CD, DA respectively. The inversion about the incircle maps A, B, C, D, Q to the midpoints A', B', C', D', Q' of segments KN, KL, LM, MN, KM respectively, also it maps P to the centroid G of the quadrilateral $KLMN$, and maps R to the reflection R' of I about G (fig. 9). Also we can obtain this point, applying to L the composition of three homotheties: centered at N with coefficient $1/2$, centered at Q' with coefficient $1/2$, and centered at I with coefficient 2. This composition maps I to the center of nine-point-circle of triangle MNK, therefore R' lies on the circle $A'D'Q'$.

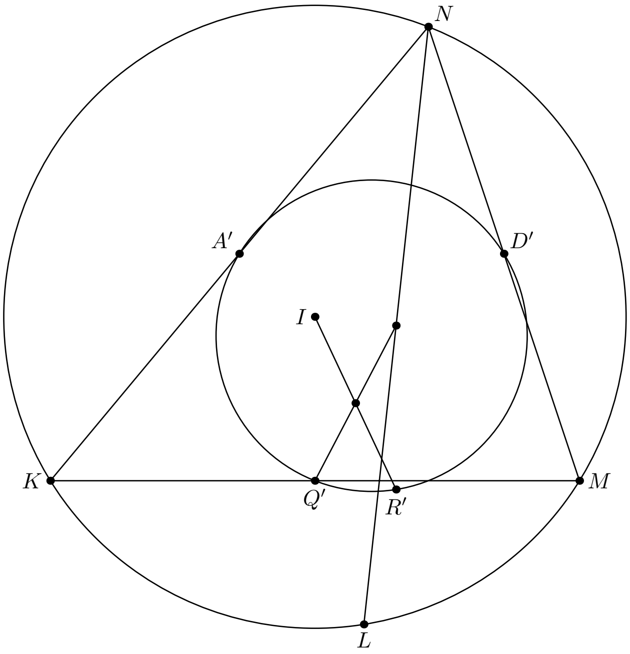

Fig. 9.

10. (8–9) (E.Volokitin) A circle ω, a point A on it, and a point B are given. Let X be an arbitrary point of ω, and T be the common point of tangents to the circle ABX at X and B. Find the locus of points T.

    Answer. The line passing through the midpoint of BC and perpendicular to BD, where C is the second common point of ω and the line AB, and D is the point of ω opposite to C.

    Solution. Let P be the common point of the tangents, Q be the reflection of B about P. Since $PQ = PB = PX$, the triangle BXQ is right-angled, and $\angle BQX = 90^\circ - \angle XBP = 90^\circ - \angle XAC = \angle DCX$ (fig. 10). Hence the triangles XQB and XCD are similar, and thus the triangles XQC and XBP are also similar. Therefore $XQ \perp BD$, which yields the answer (it is clear that any point of the line can be obtained as a common point of the tangents)

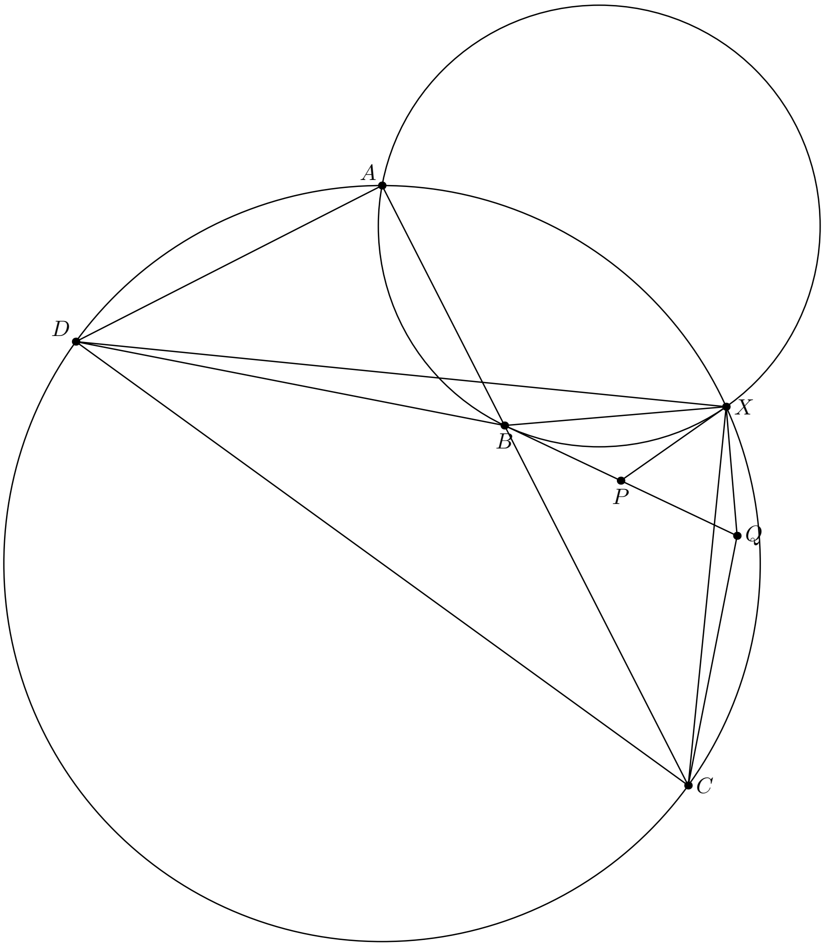

Fig. 10.

11. (8–10) (A.Mardanov) Let P and Q lie on the side AC of a triangle ABC in such a way that $PQ = AC/2$. The point B' is the reflection of B about AC. Let D and E be the points on BP and BQ such that the lines AD and CE touch the circles APB' and CQB' respectively. Prove that the circumcircle of triangle BDE touches AC.

    Solution. Let a point M be such that P is the midpoint of AM. Then Q is the midpoint of CM. Construct the circle passing through B and touching AC at M. Let BP, BQ meet this circle for the second time at points D', E' respectively (fig. 11). Since $AP^2 = AM^2 = PD' \cdot PB$, the triangles APD' and BPA are similar. Therefore $\angle D'AP = \angle PBA = \angle AB'P$, i.e. the line AD' touches the circle APB', and D' coincides with D. Similarly E' coincides with E.

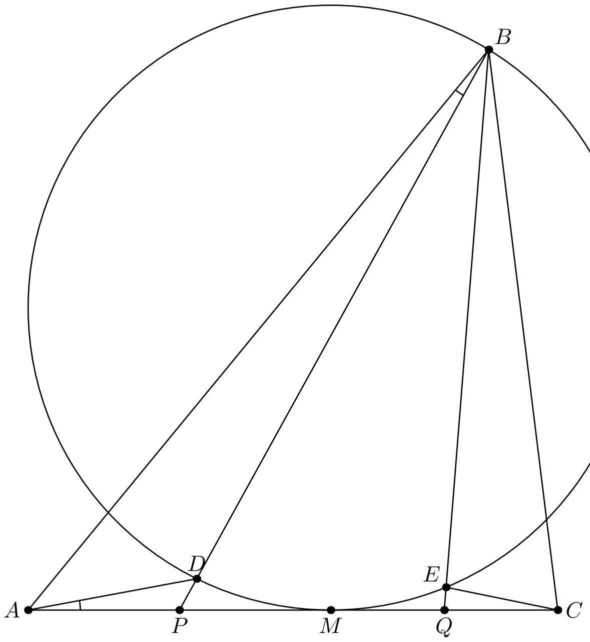

Fig. 11.

12. (8–10) (A.Tereshin) The vertices of a right-angled triangle ABC are points with integer coordinates. Its incircle centered at I touches AB, BC at points C', A' respectively. The lines AA' and CC' meet at point G. Prove that the line IG passes through some point with integer coordinates.

    First solution. Clearly the reflection D of the vertex of right-angle C about the hypothenuse AB has integer coordinates. Let us prove that IG passes through D. Let A_1, B_1, C_1 be the touching points of the incircle with BC, CA, AB respectively, A_2 be the common point of lines BC and B_1C_1, B_2 be the common point of AC and A_1C_1. then $IA_2 \perp AA_1$, $IB_2 \perp BB_1$, $IG \perp A_2B_2$. Also, since CA_1IB_1 is a square, we have $B_1B_2 = BC$, $A_1A_2 = AC$, therefore $A_2C = AB_1$, $B_2C = BA_1$ (fig. 12). And since the projections of DI to AC and BC equal AB_1 and BA_1 respectively, we obtain that $ID \perp A_2B_2$, i.e. G, I, D are collinear.

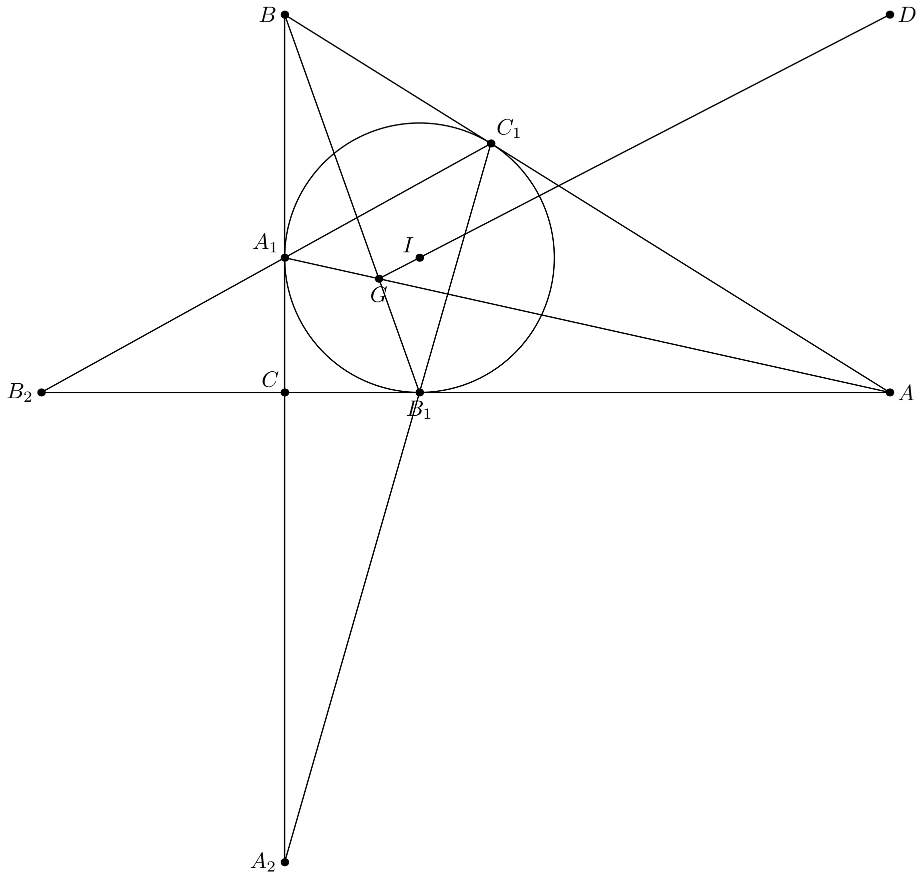

Fig. 12.

Second solution. Let U, V be the infinite points of AC, BC respectively. Applying the Pappus theorem to the triplets of points (B_1, A, U) and (A_1, B, V) we obtain that G, I, D are collinear.

13. (8–11) (D.Krokhalev) Let $A_1 \dots A_n$ be a convex polygon. The points $A_1, \dots, A_n$ in some order are vertices of two closed broken lines. What is the maximal possible ratio of their lengths?

Answer. $[n/2]$.

Example. Let the vertices of the polygon are dissected into two sets: $A_1, \dots, A_{[n/2]}$ and $A_{[n/2]+1}, \dots, A_n$ in such a way that the distances between the vertices inside each set are small, and the distance between any two vertices from different sets is close to 1. Then the vertices can be joined by a broken line with the length arbitrary close to 2, and by a broken line with the length arbitrary close to n for even n and $n-1$ for odd n.

Estimate. For even n the length of any broken-line is greater than $2d$, and less than nd, where d is the maximal distance between the vertices.

Consider an odd $n = 2k+1$. Note that two intersecting links XY and ZT can be replaced to XZ and YT, which decreases the length of the broken line. Hence the broken line with minimal length is $A_1 \dots A_n$, and the maximal length is obtained when A_i is joined with A_{i+k} and A_{i-k} (modulo n), because any two non-adjacent links of this broken line intersect. Summing n inequalities $A_iA_{i+k} + A_iA_{i-k} < P - A_{i+l}A_{i-k}$, we obtain the required boundary.

14. (9–11) (V.Konyshev) A triangle ABC ($AB < AC$) is given. Let P be a point on the ray BA such that $BP = AC$, and Q be a point on the ray CA such that $CQ = AB$. Let BB_1 and CC_1 be the perpendiculars to the line PQ. Prove that the circles (CB_1Q), (BC_1P) and the external bisector of angle BAC have a common point.

    Solution. The assumption yields that $AP = AQ$, thus PQ is perpendicular to the external bisector of angle A, and the lines BB_1, CC_1 are parallel to this bisector. Let U, V be the projections of B, C respectively to the external bisector A. From $AB = CQ$ we have $AU = CC_1$, i.e. ACC_1U is a parallelogram, $C_1U = AC = BP$, and $PUBC_1$is an isosceles trapezoid, i.e. U lies on the circle BPC_1. Similarly V lies on the circle CQB_1 (fig. 14). The powers of A about these circles are equal to $AP \cdot AB$ and $AQ \cdot AC$, i.e. their ratio equals $AB : AC = AU : AV$. Hence the second common points of these circles with the bisector coincide.

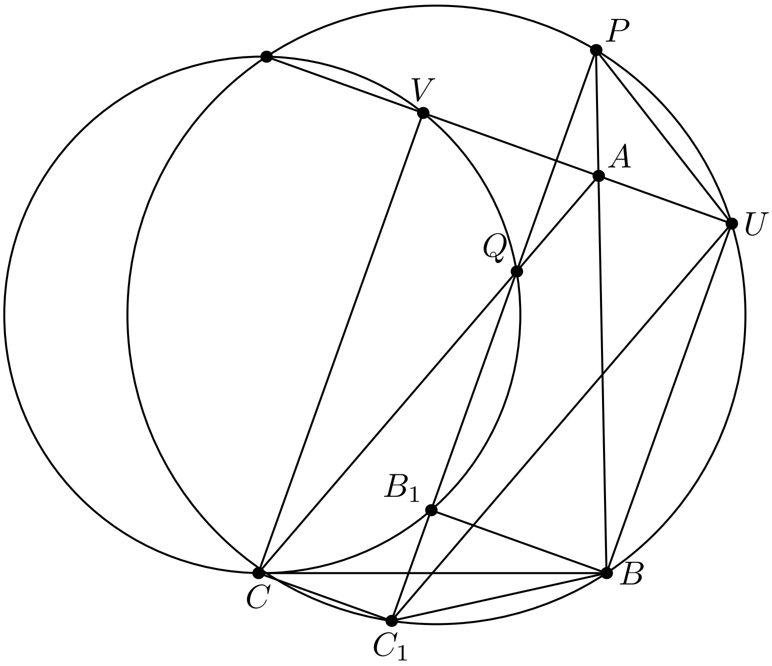

Fig. 14.

15. (9–11) (L.Emelyanov, P.Kozhevnikov) Prove that the Nagel point of a triangle lies on its incircle if and only if the bisectors of two angles meet the side of the Gergonne triangle cutting the third vertex, at two points such that the segment between them equals a half of this side.

    Solution. Let I be the incenter of ABC, and BC, CA, AB touch the incircle at points A', B', C' respectively. Suppose that the Nagel point N lies on the arc $A'B'$. Then, since the nagelian from C meets this arc at the point opposite to C', we have $S_{ABN} = 2S_{ABI}$. Since $S_{ABN} : S_{ABC} = (p - c) : p$, $S_{ABI} : S_{ABC} = c : p$, where a, b, c are the sidelengths of the triangle, and p is its semiperimeter, we obtain that $c = p - c$, i.e. the length of AB equals a quarter of the perimeter.

    Let the bisectors AI, BI meet the segment $A'B'$ at points P, Q respectively. It is known that these points lie on the medial lines parallel to BC and AC respectively (fig. 15). Hence $PB' : A'B' = BC/2 : A'C = a/2 : (p - c)$, $QA' : A'B' = b/2 : (p - c)$, and

$PQ : A'B' = c/2 : (p - c)$. Thus $PQ = A'B'/2$ also is true iff the length of AB equals a quarter of the perimeter.

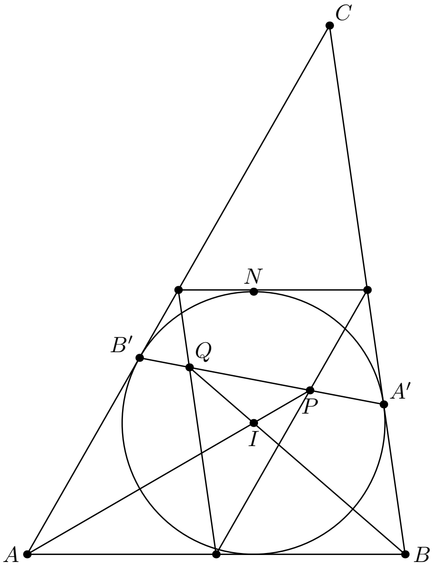

Fig. 15.

Remark. Triangles with one sidelength equal to a quarter of the perimeter have some other interesting properties. For example the incircle of such triangle touches the corresponding medial line.

16. (9–11) (M.Panov) The line passing through the common point of the diagonals of a trapezoid $ABCD$ and parallel to its bases meets the lateral side AB at point M. Let K be the projection of M to CD. Prove that KM bisects the angle AKB.

    Solution. The required assertion is equivalent to the equality $(KA, KB, KM, KC) = -1$. Let the diagonals of the trapezoid meet at point L, and the lateral sidelines meet at point S (fig. 16). Then $SA : SB = AD : BC = AL : LC = AM : MB$, therefore A, B, M, S is a harmonic quadruple.

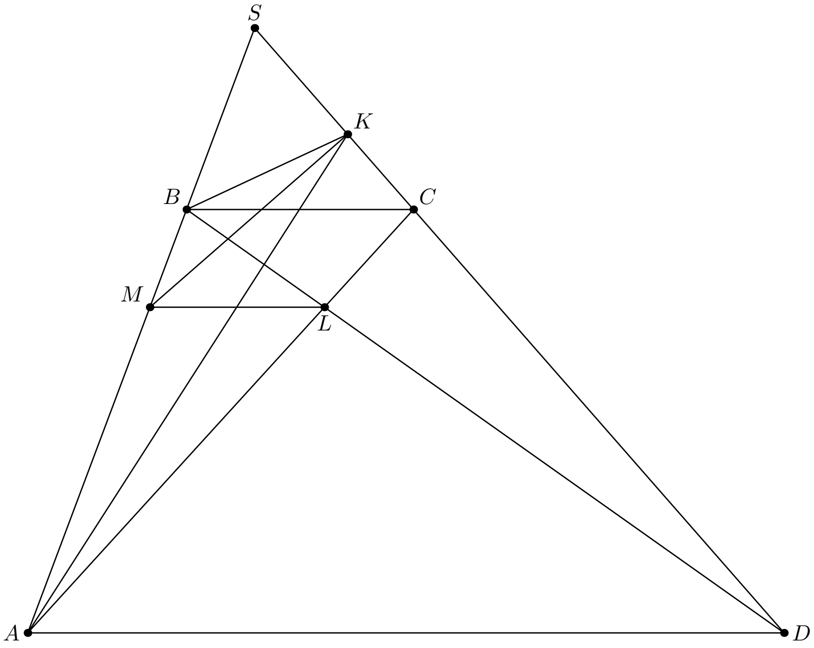

Fig. 16.

17. (9–11) (K.Belsky) Let O and H be the circumcenter and the orthocenter of an acute-angled triangle ABC. The tangents to the circumcircle of ABC at B and C meet at point T. Points X and Y lie on AB and AC in such a way that $\angle AOX = \angle AOY = \angle BTC$. Prove that $HT \perp XY$.

    Solution. Let the rays OX, OY meet the circumcircle at points U, V respectively. Then $\angle OAU = \angle TCB = \angle CAB$, therefore $\angle BAU = \angle CAO = \angle HAB$, i.e. U is the reflection H about AB. Similarly V is the reflection of H about AC (fig. 17).

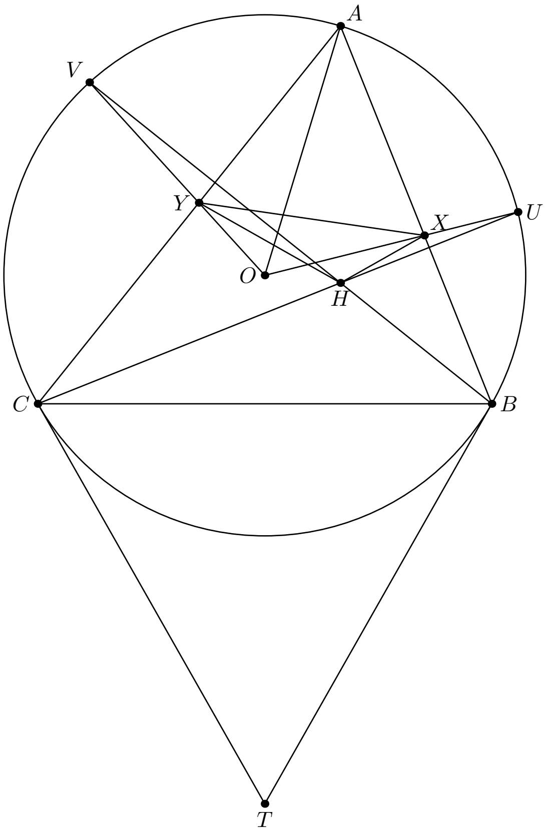

Рис. 17.

Now we can accomplish the solution by several ways.

First way. The reflection of V about the perpendicular bisector to AC is opposite to B on the circumcircle of ABC. Therefore $BT \perp BO \parallel HY$. Similarly $CT \perp HX$. Thus the triangles AXY and TBC are orthologic, and both orthology centers coicide with H, which yields the required assertion.

Second way. By symmetry we have $HX = XU = R - OX$, where R is the circumradius. Also $\angle TOY = \angle TOC + \angle COV = \angle A + 2\angle CBH = \pi - 2\angle C + \angle A$. Hence $TY^2 + HX^2 = TO^2 + OY^2 + 2TO \cdot OY \cos(2C - A) + (R - OX)^2$ or

$$TY^2 + HX^2 = TO^2 + R^2 + OX^2 + OY^2 + 2TO(OY\cos(2C - A) - OX\cos A).$$

Similarly we obtain

$$TX^2 + HY^2 = TO^2 + R^2 + OX^2 + OY^2 + 2TO(OX\cos(2B - A) - OY\cos A).$$

Applying the sines law to the triangles OAX, OAY we obtain thah $OX : R = \sin\angle OAX : \sin\angle OXA = \cos C : \cos(B - A)$ and $OY : R = \cos B : \cos(C - A)$. Therefore

$$OX : OY = \cos C\cos(C-A) : \cos B\cos(B-A) = (\cos A+\cos(2C-A)) : (\cos A+\cos(2B-A)),$$

thus $TX^2 + HY^2 = TY^2 + HX^2$, which is equivalent to the required.

18. (9–11) (K.Belsky) A point P inside a triangle ABC is given. The lines BP, CP meet the circle ABC for the second time at points E, F respectively. The circle Ω passes through

P, E and meets AC at points B_1, B_2. The lines PB_1, PB_2 meet AB at points C_1, C_2. Prove that C_1, C_2, P, F are concyclic.

Solution. When B_1 moves along AC the points B_2, C_1, C_2 move projectively. Hence it is sufficient to prove the required assertion for three positions.

Let one of points B_1, B_2 coincides with the common point of BP and AC. Then the other point is infinite, one of points C_1, C_2 coincides with B, and the second one coincides with a point Q such that $PQ \parallel AC$. Then $\angle QPF = \angle ACF = \angle ABF$, therefore P, B, F, Q are concyclic (fig.18).

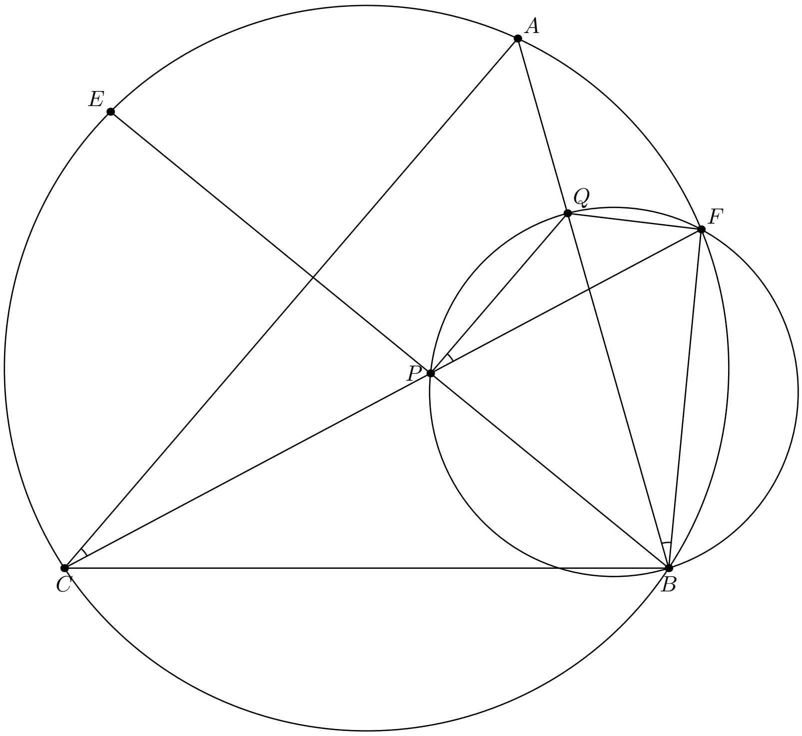

Fig. 18.

Similarly we obtain the required assertion when one of points B_1, B_2 coincides with C.

19. (10–11) (Ya.Scherbatov) The incircle of a triangle ABC centered at I touches BC, CA, AB at points A', B', C' respectively. Let A_b, A_c, B_a, C_a be the midpoints of segments $A'B$, $A'C$, $B'A$, $C'A$ respectively. The lines A_bB_a and A_cC_a meet at point P. Prove that the reflection of I about P lies on AA'.

    Solution. Let the homothety centered at I with coefficient 2 maps A_b, A_c, B_a, C_a to X', Y', X, Y respectively. The required assertion is equivalent to the perspectivity of triangles AXY and $A'X'Y'$.

    Note that $\angle BCY' = \angle CBX' = \angle CAX = \angle BAY = \pi/2$ and $AX = AY = BX' = CY' = IA_1$. Take a point Q on the altitude from A, such that $AQ = IA_1$ (fig. 19). Since $ABX'Q$, $ACY'Q$, $A'B'XQ$, and $A'C'TQ$ are parallelograms, we obtain that $X'Q \perp AY$, $Y'Q \perp AX$, $A'Q \perp XY$, $XQ \perp A'Y'$, $YQ \perp A'X'$, $AQ \perp X'Y'$, i.e. the triangles AXY and $A'X'Y'$ are orthologic and both orthology centers coincide with Q. Therefore the triangles are perspective.

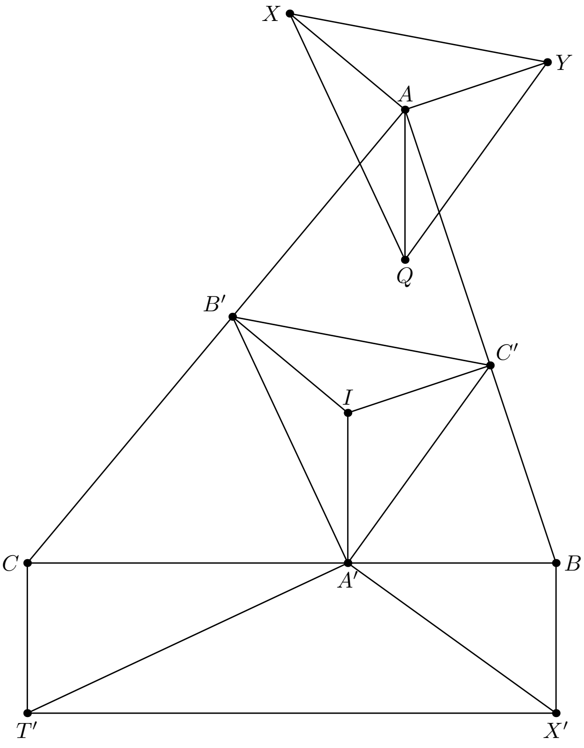

Fig. 19.

20. (10–11) (Yu.Nagumanov) The altitudes AA_1, BB_1, CC_1 of a triangle ABC meet its circumcircle for the second time at points A_2, B_2, C_2 respectively. Let A_3 be the common point of circles ABC and AB_1C_1, distinct from A; the points B_3, C_3 are defined similarly; A_4, B_4, C_4 are the feet of altitudes of triangle $A_1B_1C_1$. Prove that the lines AA_4, BB_4, CC_4, A_2A_3, B_2B_3, C_2C_3 concur.

Solution. Note that H is the incenter of $A_1B_1C_1$, thus it is also the incenter of $A_2B_2C_2$. Since B_3 lies on the circle with diameter BH, B_3H passes through the midpoint of the major arc A_2C_2. Then B_3 is the touching point of the mixtilinear circle. Then A_2A_3, B_2B_3, C_2C_3 meet at the center of positive homothety mapping the incircle of $A_2B_2C_2$ to its circumcircle. Note that the line joining the homothety center and the midpoint of the arc passes through the touching point of the incircle. Consider the homothety with center H and coefficient 2. It maps B to the excenter I_B of $A_2B_2C_2$, maps the touching point to the reflection of H about A_2C_2, and maps B_4 to the foot B' of the altitude.

Since B_2, the foot of the bisector of $\angle A_2B_2C_2$, H, and I_B form a harmonic quadruple, and $\angle B_2B'A_2 = \pi/2$, we obtain that $B'A_2$ bisects the angle $HB'I_B$ (fig. 20). Hence BB_4 passes through the touching point of A_2C_2 with the incircle, q.e,d.

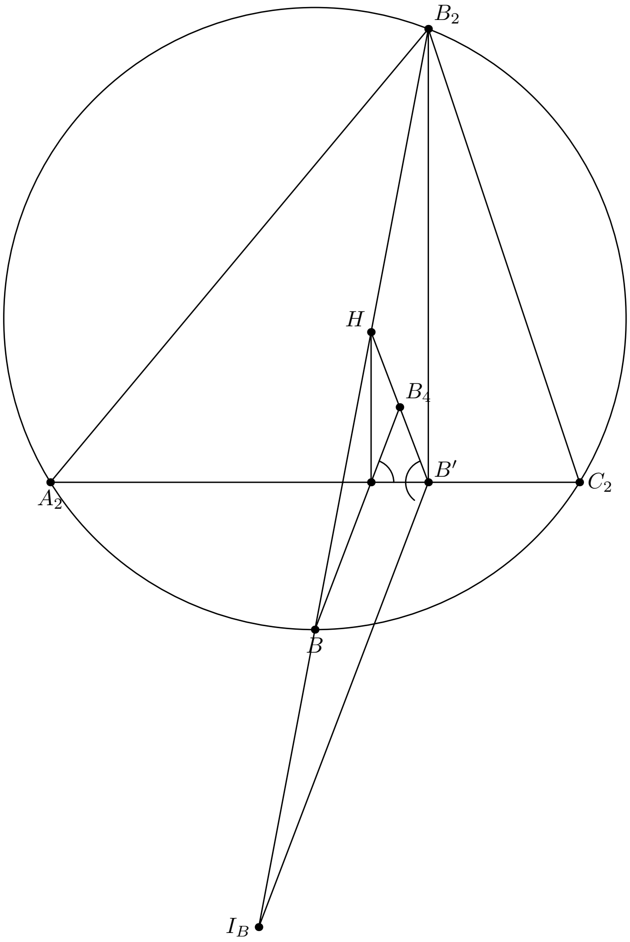

Fig. 20.

21. (10–11) (Tran Quang Hung) Let ABC be a triangle with $\angle A = 2\pi/3$; P be an arbitrary point inside this triangle lying on the bisector of angle A; the lines BP, CP meet AC, AB at points E, F respectively; D be an arbitrary point on the side BC; the lines DE, DF meet PC, PB at points M, N respectively. Find the value of angle MAN.

 Answer. 60°.

 Solution. Denote by $\mathcal{R}$ the rotation around A mapping AC to AP. Projecting from D the line CP to BP we obtain the equality $(P, C, F, M) = (P, B, N, E)$. On the other hand $(AP, AC, AF, AM) = (\mathcal{R}(AP), \mathcal{R}(AC), \mathcal{R}(AF), \mathcal{R}(AM)) = (AB, AP, AE, \mathcal{R}(AM)) = (B, P, N', E)$, where N' is the common point of BE and $\mathcal{R}(AM)$. therefore $N' = N$ and $\angle MAN = 60^\circ$ (fig. 21).

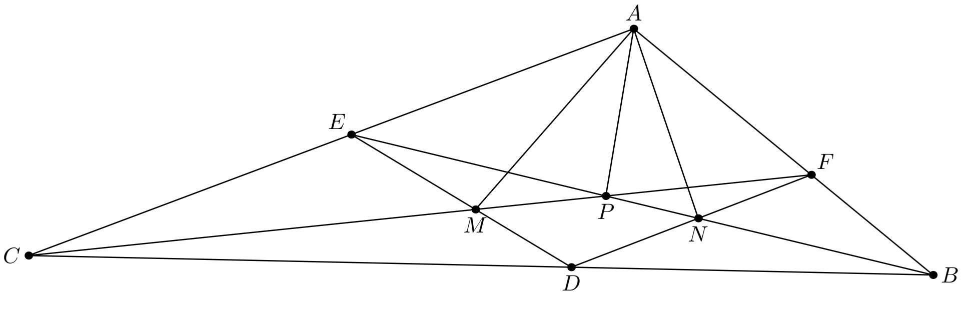

Fig. 21.

22. (10–11) (S.Chuev) The incircle of a triangle ABC touches BC at point D. Let F be the Feuerbach point, H be the projection of A to DF. Prove that $FH : DF = 1 : 2$.

Solution. Let I be the incenter of ABC, N be the point opposite to D, M be the midpoint of AI. Since the projections of N and I to DF coincide with F and the midpoint of DF respectively, we have to prove that M, N, F are collinear, i.e. $\angle MFD = \pi/2$. Let K, L, P be the midpoints of AB, BI, IC respectively (fig. 22). It is known that F lies on the nine-points-circles of triangles ABI and BCI, therefore $\angle MFD = \angle MFL + \angle LFD = \pi - \angle LKM + \angle LPD = \pi - \angle AIB + \angle ICB = \pi - (\pi + \angle ACB)/2 + \angle ICB = \pi/2$.

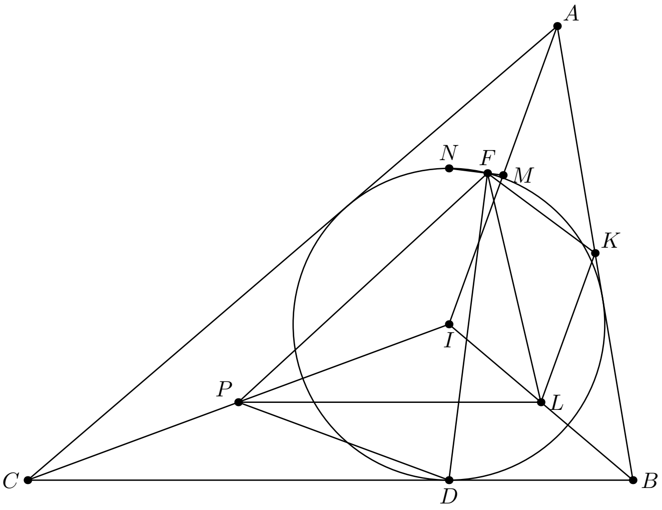

Fig. 22.

23. (10–11) (V.Shelomovsky) A triangle ABC and a point P in the plane are given. The lines AP, BP, CP meet for the second time the circle ABC at points A', B', C' respectively. Find the locus of points Q such that the common points of QA' with BC, QB' with AC, and QC' with AB are collinear.

Answer. The circumcircle of ABC and the polar of P about it.

Solution. Let the lines $A'Q$ and BC meet at point X, the lines $B'Q$ and AC meet at point Y, and the lines $C'Q$ and AB meet at point Z. It is easy to see that the ratios $BX : XC$, $CY : YA$, $AZ : ZB$ are fractionally linear functions of the coordinates (x, y) of Q. Thus the collinearity of X, Y, Z gives an equality of degree 3 for (x, y). Hence it is sufficient to prove that any point of the circumcircle or the polar of P lies on the required locus.

Let Q lie on the circumcircle of ABC. Applying the Pascal theorem to the hexagon $ACBB'QA'$ we obtain that the line XY passes through P (fig. 23). Similarly Z lies on this line.

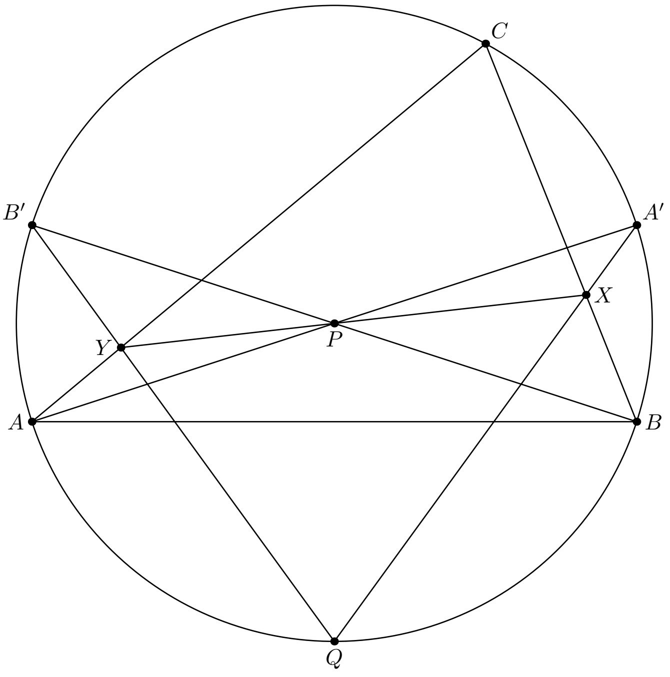

Fig. 23.

Let now AC and $A'C'$ meet at point U, BC and $B'C'$ meet at point V, and Q lie on the polar UV of P. Since the triangles ABC and $A'B'C'$ are perspective, they have a common inconic κ. Applying the Brianshon theorem to the hexagon $A'UYXB'V$we obtain that XY touches κ. Then Z also lies on this line.

24. (11) (A.Tereshin) The insphere of a tetrahedron $ABCD$ touches the face ABC at its orthocenter H and touches the faces ABD, ACD at points P, Q respectively. The lines DP, DQ meet the plane ABC at points X, Y. Prove that $\angle XHY = 2\angle XAY$.

    Solution. The cone with vertex D circumscribed around the insphere meets the plane ABC by an ellipse with focus H, touching AB, AC at points X, Y. The second focus of this ellipse is the circumcenter O of triangle ABC. Since the reflections U, V of H about AB and AC lie on the circumcircle, and X, Y lie on OU, OV respectively, we have $\angle XHY = \angle UHV - \angle UHX - \angle VHY = \pi - \angle CAB - \angle HCO - \angle HBO = 2\angle YAX$ (fig. 24).

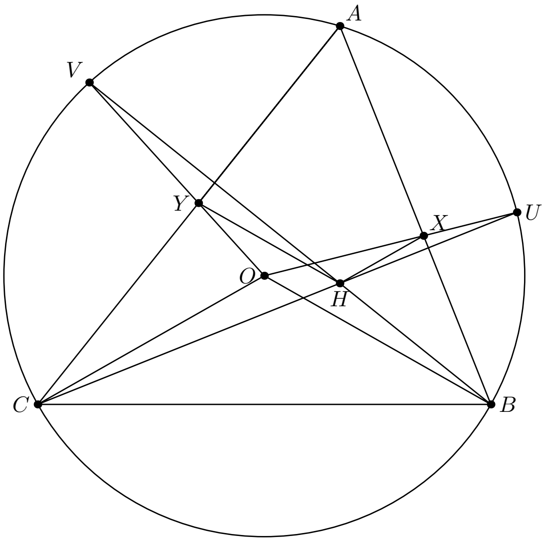

Рис. 24.